October Math Gems

PROBLEM OF THE WEEK 13

§1 Problems

Problem 1.1. If the line $kx + 4y = 6$ passes through the point of intersection of the two lines $2x + 3y = 4$ and $3x + 4y = 5$

Solution. Solving the equations $2x + 3y = 4$ and $3x + 4y = 5$ we get that $x = -1; y = 2$ which means that the point of intersection is (-1,2). By using this point in $kx + 4y = 6$ we get that $k = 2$ □

Problem 1.2. If p_1 and p_2 are length of the perpendicular from the origin on the two lines given by $x \sec A + y \csc A = m$ and $x \cos A + y \sin A = m \cos 2A$ then $4(p_1)^2 + (p_2)^2 = ...$ with respect to m

Solution. $x \sec A + y \csc A = m; ==> x \sin A + y \cos A = m \sin A \cos A = \frac{m}{2} \sin 2A$
$x(2 \sin A) + y(2 \cos A) - m \sin 2A = 0 \therefore\ p_1 = \frac{|-m \sin 2A|}{\sqrt{(2 \sin A)^2 (2 \cos A)^2}} = |\frac{m}{2} \sin 2A|$

$2p_1 = |\frac{m}{2} \sin 2A|; 4(p_1)^2 = m^{2 \sin^2 A}$

Similarly, $(p_2)^2 = m^2 \cos^2 2A$. Then $4(p_1)^2 + (p_2)^2 = m$ □

Problem 1.3. If (4-,5) is one vertex and $7x - y + 8 = 0$ is one diagonal of a square, then the equation of the other diagonal is....

Solution. Given that $7x - y + 8 = 0$ (1) is one diagonal of a square
WE know that square's diagonals are perpendicular
Line perpendicular to (1) is given by $x + 7y = c$
The given point doesn't satisfy (1), so it must be the vertex of the other diagonal
$c = 31$. Then the other diagonal's equation $x + 7y = 31$ □

Problem 1.4. A triangle has vertices $A(0, b); B(0, 0); C(a, 0)$. If its median AD and BE are mutually perpendicular then $a =$ .... with respect to b

Solution. D is the mid point of $BC = (\frac{a}{2}, 0), E = (\frac{a}{2}, \frac{b}{2})$

AD slope * BE slope = -1

$\frac{0-b}{\frac{a}{2}-0} * \frac{\frac{b}{2}-0}{\frac{a}{2}-0} = -1 ==> \frac{b^2}{2} = \frac{a^2}{4}; a = +or - b\sqrt{2}$ □

Problem 1.5. Points (3,3),(h,0),(0,k) are collinear and $\frac{a}{h} + \frac{b}{k} = \frac{1}{3}$. Then what is the value of a, b

Solution. The points are collinear $\begin{vmatrix} 1 & 4 & 1 \\ 2 & 6 & 3 \\ 2 & 10 & 3 \end{vmatrix}$ =0

$3(0-k)-3(h-0)+1(hk-0)=0$

$3j+3k=hk ==> \frac{1}{h}+\frac{1}{k}=\frac{1}{3}$ Then $a=b=1$ □

Problem 1.6. Let $p(x)$ be a cubic polynomial with zeros α,β,γ. If $\frac{p(\frac{1}{2})+p(\frac{-1}{2})}{p(0)}=100$ find $\sqrt{\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}}$

Solution. Let $p(x)=ax^3+bx^2+cx+d$
$p(\frac{1}{2})+p(\frac{-1}{2})=2b(\frac{1}{4}+2d$ and $p(0)=d$

$==> \frac{2\frac{b}{4}+2d}{d}=100$ which means $\frac{b}{d}=196$

$\alpha+\beta+\gamma=\frac{-b}{a},\alpha\beta\gamma=\frac{-d}{a}$

Hence, $\sqrt{\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}}=\sqrt{\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}}=\sqrt{196}=14$ □

Problem 1.7. Find the value of a and b so that $x^4+x^3+8x^2+ax+b$ is divisible by x^2+1

Solution. First, we will divide $x^4+x^3+8x^2+ax+b$ by x^2+1. We will get a remainder of $(a-1)x+(b-7)$ which must equal 0
$(a-1)=0$ or $(b-7)=0$ Hence, $a=1,b=7$ □

Problem 1.8. Consider the equation $x^3-(1+\cos\theta+\sin\theta)x^2+(\cos\theta\sin\theta+\cos\theta+\sin\theta)x-\sin\theta\cos gq=0$ The roots of equation are x_1,x_2,x_3 Find the value of $(x_1)^2+(x_2)^2+(x_3)^2$

Solution. $(x_1)^2+(x_2)^2+(x_3)^2=(x_1)^2+(x_2)^2+(x_3)^2+2(x_1x_2+x_2x_3+x_1x_3)$
$x_1+x_2+x_3=1+\cos\theta+\sin\theta$
$(x_1x_2+x_2x_3+x_1x_3)=\cos\theta\sin\theta+\cos\theta+\sin\theta$
$(1+\cos\theta+\sin\theta)^2=(x_1)^2+(x_2)^2+(x_3)^2+2(\cos\theta\sin\theta+\cos\theta+\sin\theta)$
$(x_1)^2+(x_2)^2+(x_3)^2=1+\cos^2\theta+\sin^2\theta=2$ □

Problem 1.9. The inverse function of the function $f(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}}=$

Solution. $y=\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{e^{2x}-1}{e^{2x}+1}$

$y(e^{2x}+1)=(e^{2x}-1)$

$y+1=e^{2x}(1-y)$ Then $e^{2x}=\frac{1+y}{1-y}$

Taking ln for both sides $2x=\ln(\frac{1+y}{1-y})$

$f^{-1}=\frac{1}{2}\ln(\frac{1+x}{1-x})$ □

Problem 1.10. Let $f(x)=\frac{ax+b}{cx+d}$ then $fof(x)=x$ Find d

Solution. Given that $f(x) = \frac{ax+b}{cx+d}$

$f(f(x)) = \frac{a(ax+b)+b(cx+d)}{c(ax+b)+d(cx+d)} = x$

$(a^2 + bc)x + (ab + bd) = (ac + cd)x^2 + (bc + d^2)x$

matching the coefficients of x, we get $a^2 + bc = bc + d^2$ then $d = +or - a$

matching the coefficients of x^2, we get $ac + cd = 0$ then $d = -a$

matching the coefficients of constant, we get $ab + bd = 0$ then $d = -a$ □

Problem 1.11. Let $f(x) = \frac{ax}{x+1}$ for x not equal -1. Then what is the value of a if $f(f(x)) = x$

Solution. $f(f(x)) = \frac{a(\frac{ax}{x+1})}{\frac{ax}{x+1}+1} = \frac{a^2x}{ax+x+1}$

$\frac{a^2x}{ax+x+1} = x$

$a^2 = ax + x + 1$
$a^2 - ax - (x + 1) = 0$
$a^2 - ax - a + a - (x + 1) = 0$
$a(a - (x + 1)) + (a - (x + 1)) = 0$
$(a + 1)(a - (x + 1))$ Then $a = -1$ □

Problem 1.12. The function f is one-to-one and the sum of all the intercepts of the graph is 5. The sum of all intercepts of the graph $y = f^{-1}(x)$ is

Solution. Since the function is one-to-one there exist only one x-intercept and only one y-intercept. We know that x-intercept of $f(x)$ = y-intercept of $f^{-1}(x)$ and visa verse. Hence, the answer is 5 □

Problem 1.13. $\frac{f(x)}{g(x)} = x - 2$ with remainder $4 - 2x$ Find $g(x)$ if $f(x) = x^3 - 3x^2 + x + 2$

Solution. $f(x) = g(x) * (x - 2) + 4 - 2x$
$g(x) = \frac{x^3-3x^2+x+2}{x-2}$

using long or synthetic division $g(x) = x^2 - x + 1$ □

Problem 1.14. The expansion $\frac{1}{\sqrt{4x+1}}((\frac{1+\sqrt{4x+1}}{2})^2 - (\frac{1-\sqrt{4x-1}}{2})^2)$ is a polynomial of x degree =

Solution. $\frac{1}{\sqrt{4x+1}}((\frac{1+\sqrt{4x+1}}{2})^2 - (\frac{1-\sqrt{4x-1}}{2})^2) = \frac{1}{2^2\sqrt{4x+1}}((1 + \sqrt{4x + 1})^2 - (1 - \sqrt{4x + 1})^2)$

$\frac{4\sqrt{4x+1}}{4\sqrt{4x+1}} = 1$ □

Problem 1.15. b, a are zeros of $h(x) = 3x^2 - 6x + 12$ find the value of $a^{-1} + b^{-1}$

Solution. $h(x) = 3x^2 - 6x + 12,\ a + b = 2, ab = 4$
$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{2}{4} = \frac{1}{2}$ □

Problem 1.16. If α, β, γ are zeros of polynomial $x^3 - x - 1$, then the value of $\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}$

Solution. comparing the equation with $ax^3 + bx^2 + cx + d, a = 1, b = 0, c = -1, d = -1$, $\alpha + \beta + \gamma = \frac{-b}{a} = 0, \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = -1, \alpha\beta\gamma = \frac{d}{a} = 1$

$\frac{(1+\alpha)(1-\beta)(1-\gamma)+(1+\beta)(1-\alpha)(1-\gamma)+(1+\gamma)(1-\beta)(1-\alpha)}{(1-\alpha)(1-\beta)(1-\gamma)} = \frac{3+3\alpha\beta\gamma-(\alpha+\beta+\gamma)-(\alpha\beta+\beta\gamma+\gamma\alpha)}{1-(\alpha+\beta+\gamma)+(\alpha\beta+\beta\gamma+\gamma\alpha)+\alpha\beta\gamma}$

$= \frac{3+3+1}{1} = 7$ □

Problem 1.17. suppose $f(x) = ax + b$ and $g(x) = bx + a$ where a, b are positive integers. If $f(g(50)) - g(f(50)) = 28$ then the product of ab cam have the value of

Solution. $f(g(x)) = abx + a^2 + b, g(f(x)) = abx + b^2 + b$
$f(g(x)) - g(f(x)) = (a-b)(a+b-1) = 4*7$ if $a-b = 4, a+b = 8$ then,$a = 6, b = 2, ab = 12$
if $f(g(x)) - g(f(x)) = (a-b)(a+b-1) = 1*28$ then$a - b = 1, a + b = 28, a = 15, b = 14, ab = 210$ □

Problem 1.18. If $f(x) = px + q$ and $f(f(f(x))) = 8x + 21$ where p, q are real number, then $p + q =$

Solution. $f(f(f(x))) = p^3 + q(p(p+1)+1) + q = 8x + 21$ by comparing the co-offients, $p^3 = 8, p = 2$
$q(2(3) + 1) = 21, q = 3, p + q = 5$ □

Problem 1.19. The sum of n terms of two arithmetic series are in the ratio of $2n + 3 : 6n + 5$ then the ratio for their $13^{th} termsis$

Solution. Let a_1, a_2 be the first terms and d_1, d_2 be the common difference at the given A.Ps, n term$S_n = \frac{n}{2}(2a_1 + (n-1)d_1)$
$S_n = \frac{n}{2}(2a_2 + (n-1)d_2)$
$\frac{S_n}{S_n} = \frac{2n+3}{6n+5} = \frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2}$
the ratio between 13^{th} term is$\frac{a_1+12d_1}{a_2+12d_2} = \frac{2a_1+(25-1)d_1}{2a_2+(25-1)d_2} = \frac{2*25+3}{6*25+5} = \frac{53}{155}$ □

Problem 1.20. The p^{th} term of an arithmetic progression is q and the q^{th} term is p the the 10^{th} term= (with respect to p, q)

Solution. Let a be the first term and d be the common difference, $t_p = q, a + (p-1)d = q, t_q = p(i), a + (q-1)d = p(ii)$ subtracting (ii)-(i), we get that $d = -1$, substituting $d = -1$ in (i), we get that $a = p + q - 1$
for the 10^{th} term, $t_10 = a + 9d = p + q - 10$ □
